

Sliding rings and spinning holes

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ABSTRACT: We construct smooth supergravity solutions describing a BPS black ring with a BPS black hole centered at an arbitrary distance above the ring. We find that as one moves the black hole the entropy of the ring remains constant, but the angular momentum coming from the supergravity fluxes changes. Our solutions also show that in order to merge a BPS black ring with a BPS black hole one has to increase one of the angular momenta of the ring, and that the result of the merger is always a BMPV black hole. We also find a class of mergers that are thermodynamically reversible, and comment on their physics.

KEYWORDS: Black Holes, Black Holes in String Theory, D-branes.

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1. Introduction

An intriguing feature of BPS black holes in five dimensions is the fact that their two angular momenta have to be equal [1, 2]. The recent conjecture [3, 4] and discovery [5–9] of BPS black rings has shown that there also exist objects with unequal angular momenta, with horizon topology $S^2 \times S^1$. Since black rings and black holes are mutually BPS, they can be placed at arbitrary positions relative to one another and can even be merged.

It is a very interesting problem to investigate the merging process of a BPS black ring with a BPS black hole and to determine the end result. It is intuitively clear that the merger can either produce a BPS black hole or a BPS black ring: Pushing a tiny black ring into a vast black hole, or vice versa, should not change the horizon topology of the vast object. This raises an interesting conundrum when one imagines a small black ring being merged with a large black hole: When the black ring and the black hole are widely separated, the two angular momenta of the ring are different from each other. Hence, the black hole that one might expect to result from the merger would, naively, have different angular momenta [3], and contradict theorems that claim that BPS five-dimensional black holes must have equal angular momenta [10].

Another thought-provoking question is whether one can overspin a BMPV black hole by dumping into it a black ring with more angular momentum than a black hole can have. A similar gedanken experiment involving merging a BMPV black hole and a two-charge supertube has been investigated in [3], and more thoroughly in [11] using the Born-Infeld action of the supertube. While this Born-Infeld analysis yields very important insights into the merger process (like the fact that the angular momentum in the plane perpendicular to the ring depends on the distance between the ring and black hole [11]), it is still only perturbative and is limited to two-charge configurations. It also does not give too much information about the back-reaction, the behavior of horizons, or the angular momenta coming from fluxes.

It is also interesting to study the process in which a three-charge supertube (zero-entropy black ring) merges with a black hole. If one combines two maximally-spinning, BPS black holes with charges, $Y^{(i)}$, and angular momenta, $J^{(i)} = (Y^{(i)})^{3/2}$, where $i = 1, 2$ labels the black hole, the resulting BMPV black hole satisfies the strict inequality $(J^{(1)} + J^{(2)}) < (Y^{(1)} + Y^{(2)})^{3/2}$. That is, the merger process is strictly irreversible in that there is an increase in the horizon area. We find that the corresponding process for black rings is “softer,” in that there is a reversible process in which a certain family of three-charge supertubes can be added to a maximally rotating BPS black hole. In this process, the entropy remains constant if the three-charge supertube exactly grazes the horizon. If three-charge supertubes are indeed microstates of black holes,¹ then there should be processes by which they can be added reversibly to another zero-entropy system. We indeed find that this is possible, and determine the precise reversible process.

In this paper we give a comprehensive analysis of the merger problem by constructing full supergravity solutions that contain a (three-charge) BPS black ring and a (three-charge) BPS black hole, where the black hole is at an arbitrary distance above the center of the ring so that the solution still has a $U(1)$ isometry, corresponding to rotation in the plane of the ring. Our solutions are much more complicated than the $U(1) \times U(1)$ invariant solutions describing concentric black rings and black holes [8, 6, 9] and reduce to those when the black hole is moved to the center of the ring.

As explained in [6, 14], and proved in [15, 16], black-ring solutions have microscopic charges and angular momenta different from those measured at infinity because of charges and angular momenta dissolved in fluxes. We find that as one changes the location of the black hole, the angular momentum in the plane of the ring remains fixed, but the other angular momentum changes. The varying part of the angular momentum precisely depends on the product of the magnetic charges of the ring and the electric charges of the black hole. As expected, this component of the angular momentum increases as the black ring and the black hole approach one another.

If the black hole charges are sufficiently large, then the black ring and the black hole merge. When this happens, we find that in order to bring the black ring up to the black hole horizon one must increase the angular momentum in the plane perpendicular to the

¹This has been proposed in [3, 12]; a more general discussion of this framework can be found in [13] and references therein

ring, so that, right before the merger, this component of the angular momentum is exactly equal to the angular momentum in the plane of the ring. Thus, our analysis shows that the result of such an axially symmetric merger of a BMPV black hole and a BPS black ring is always another BMPV black hole.

We have also computed the general expression for the entropy of the black ring and black hole, and since it depends on eleven parameters, a general analysis is rather complicated. We therefore consider a reduced (five-parameter), but representative, sub-class of black holes and black rings. We show that if the black hole has sufficient charge for the merger to take place, then the resulting black hole has a total area at least as large as the total horizon area of the original, widely separated, black hole and black ring. We therefore show, at least for this sub-class, that the area-increase theorem is respected, and therefore that one cannot “over-spin” the black hole by pushing a black ring into it.

In section 2 we re-write the $U(1) \times U(1)$ invariant solution that describes a black ring with a concentric black hole, and we carefully identify the black-ring microscopic charges and their relation to the charges of the solution. In section 3 we solve the equations that underlie supersymmetric solutions in five dimensions [6, 17, 18], by using the linear algorithm discovered in [6], and we find the exact solution corresponding to a black hole at an arbitrary distance away from the black ring center. Those interested only in the solutions can find them summarized in section 4. While our solutions are constructed in M theory compactified on T^6 , they can be trivially extended to any $U(1)^N$ supergravity in five dimensions. In section 5 we find the charges and angular momenta of the solutions while, in section 6, we analyze black-ring black-hole mergers, as well as the possibility of overspinning a BMPV black hole using black rings. Finally, in section 7 we present conclusions and suggestions for future research.

2. A black hole at the center of the ring

As shown in [18, 6], an M-theory background that preserves the same supersymmetries as three orthogonal M2-branes can be written as:

$$\begin{aligned}
 ds_{11}^2 = & - \left(\frac{1}{Z_1 Z_2 Z_3} \right)^{2/3} (dt + k)^2 + (Z_1 Z_2 Z_3)^{1/3} h_{mn} dx^m dx^n \\
 & + \left(\frac{Z_2 Z_3}{Z_1^2} \right)^{1/3} (dx_1^2 + dx_2^2) + \left(\frac{Z_1 Z_3}{Z_2^2} \right)^{1/3} (dx_3^2 + dx_4^2) + \left(\frac{Z_1 Z_2}{Z_3^2} \right)^{1/3} (dx_5^2 + dx_6^2), \\
 \mathcal{A} = & A^1 \wedge dx_1 \wedge dx_2 + A^2 \wedge dx_3 \wedge dx_4 + A^3 \wedge dx_5 \wedge dx_6,
 \end{aligned}
 \tag{2.1}$$

where A^I and k are one-forms in the five-dimensional space transverse to the T^6 . The metric h_{mn} can be any four-dimensional hyper-Kähler metric, but in this paper we focus on black rings and black holes in $\mathbb{R}^{(4,1)}$, so we take this space to be \mathbb{R}^4 . When written in terms of the “dipole field strengths,” Θ^I ,

$$\Theta^I \equiv dA^I + d\left(\frac{dt + k}{Z_I} \right),
 \tag{2.2}$$

the BPS equations simplify to² [18, 6]:

$$\begin{aligned} \Theta^I &= \star_4 \Theta^I \\ \nabla^2 Z_I &= \frac{1}{2} C_{IJK} \star_4 (\Theta^J \wedge \Theta^K) \\ dk + \star_4 dk &= Z_I \Theta^I, \end{aligned} \tag{2.3}$$

where \star_4 is the Hodge dual on \mathbb{R}^4 , and for M theory on T^6 , $C_{IJK} = |\epsilon_{IJK}|$. In addition to the M2 branes, this solution contains three sets of M5 branes that wrap four-dimensional tori in the 4567, 2367 and 2345 directions respectively, as well as a closed curve in \mathbb{R}^4 . This curve describes a black-ring profile.

In principle, one can solve these equations to find the solution for an arbitrary distribution of black holes and black rings. This is done in three steps [6]. One first solves for the self-dual field strengths, Θ^I , sourced by the M5 branes. The second step is to find the harmonic functions sourced by the actual M2 branes present in the solution, and by the M2 charge coming from the supergravity fields ($C_{IJK} \star_4 (\Theta^J \wedge \Theta^K)$). The third step is to solve for the angular momentum vector, k .

To describe the round black ring it is natural to think of the spatial \mathbb{R}^4 as $\mathbb{R}^2 \times \mathbb{R}^2$ with spherical polar coordinates (z, ϕ) and (r, ψ) in which the metric becomes:

$$d\vec{y} \cdot d\vec{y} = (dz^2 + z^2 d\psi^2) + (dr^2 + r^2 d\phi^2). \tag{2.4}$$

One then locates the ring at $r = 0$ and $z = R$. However, it greatly simplifies calculations if one introduces a coordinate system that makes the dipole fields, Θ^I , very simple. Indeed, it is conventional to use the coordinates [5]:

$$x = -\frac{z^2 + r^2 - R^2}{\sqrt{((z - R)^2 + r^2)((z + R)^2 + r^2)}}, \quad y = -\frac{z^2 + r^2 + R^2}{\sqrt{((z - R)^2 + r^2)((z + R)^2 + r^2)}}, \tag{2.5}$$

in which one has $-1 \leq x \leq 1$, $-\infty < y \leq -1$. In these coordinates, the metric on \mathbb{R}^4 becomes:

$$ds_{\mathbb{R}^4}^2 = \frac{R^2}{(x - y)^2} \left(\frac{dy^2}{y^2 - 1} + (y^2 - 1) d\psi^2 + \frac{dx^2}{1 - x^2} + (1 - x^2) d\phi^2 \right). \tag{2.6}$$

In this system the ring is located at $y = -\infty$, while spatial infinity is at $x \rightarrow -1, y \rightarrow -1$ with $(x + 1)/(y + 1)$ finite.

Since the black hole does not contain any M5 brane dipole moments, the Θ^J are the same as those of a pure ring³:

$$\Theta^J = 2q^J (dx \wedge d\phi - dy \wedge d\psi). \tag{2.7}$$

The harmonic functions are also given by the simple superposition of the ring harmonic functions and the harmonic functions, H_I , of the black hole

$$Z_I = 1 + \frac{\bar{Q}_I}{R} (x - y) - \frac{2C_{IJK} q^J q^K}{R^2} (x^2 - y^2) + H_I \tag{2.8}$$

²Note that the field strengths used here have a different normalization from those of [6].

³In the metric (2.6) we take $\epsilon^{y\psi\phi} = +1$.

For the present we consider the solution with the black hole at the center of the ring and so one has [6, 9]:

$$H_I = -\frac{Y_I}{R^2} \frac{x-y}{x+y}, \quad (2.9)$$

where Y_I are the charges of the black hole. For this configuration the angular momentum components are:

$$\begin{aligned} k_\psi &= (y^2 - 1) \left(\frac{C}{3} (x+y) + \frac{B}{2} - \frac{D}{R^2(x+y)} + \frac{K}{R^2(x+y)^2} \right) - A(y+1), \\ k_\phi &= (x^2 - 1) \left(\frac{C}{3} (x+y) + \frac{B}{2} - \frac{D}{R^2(x+y)} + \frac{K}{R^2(x+y)^2} \right). \end{aligned} \quad (2.10)$$

where K represents the angular momentum of the BMPV black hole and

$$\begin{aligned} A &\equiv 2\left(\sum q^I\right), & B &\equiv \frac{2}{R}(Q_I q^I), \\ C &\equiv -\frac{8 C_{IJK} q^I q^J q^K}{R^2}, & D &\equiv 2 Y_I q^I \end{aligned} \quad (2.11)$$

The relation between the quantized ring and black-hole charges and the parameters appearing in the solution are:

$$\bar{Q}_I = \frac{\bar{N}_I l_p^6}{2L^4 R}, \quad q^I = \frac{n^I l_p^3}{4L^2}, \quad Y_I = \frac{N_I^{\text{BH}} l_p^6}{L^4}, \quad K = \frac{J^{\text{BMPV}} l_p^6}{L^6}. \quad (2.12)$$

where L is the radius of the circles that make up the T^6 (so that $V_6 = (2\pi L)^6$). The asymptotic charges, N_I , of the solution are the sum of the microscopic charges on the black ring, \bar{N}_I , the charges of the black hole, N_I^{BH} , and the charges dissolved in fluxes:

$$N_I = \bar{N}_I + N_I^{\text{BH}} + \frac{1}{2} C_{IJK} n^J n^K. \quad (2.13)$$

The angular momenta of this solution are:

$$\begin{aligned} J_1 &= J_\Delta + \left(\frac{1}{6} C_{IJK} n^I n^J n^K + \frac{1}{2} \bar{N}_I n^I + N_I^{\text{BH}} n^I + J^{\text{BMPV}} \right) \\ J_2 &= -\left(\frac{1}{6} C_{IJK} n^I n^J n^K + \frac{1}{2} \bar{N}_I n^I + N_I^{\text{BH}} n^I + J^{\text{BMPV}} \right), \end{aligned} \quad (2.14)$$

where

$$J_\Delta \equiv \frac{R^2 L^4}{l_p^6} \left(\sum n^I \right). \quad (2.15)$$

The entropy of the ring is:

$$S = \frac{2\pi A}{\kappa_{11}^2} = \pi \sqrt{\mathcal{M}} \quad (2.16)$$

where

$$\begin{aligned} \mathcal{M} &\equiv 2n^1 n^2 \bar{N}_1 \bar{N}_2 + 2n^1 n^3 \bar{N}_1 \bar{N}_3 + 2n^2 n^3 \bar{N}_2 \bar{N}_3 - (n^1 \bar{N}_1)^2 - (n^2 \bar{N}_2)^2 - (n^3 \bar{N}_3)^2 \\ &\quad - 4n^1 n^2 n^3 (J_\Delta + n^I N_I^{\text{BH}}). \end{aligned} \quad (2.17)$$

Since the entropy of the black ring is the square root of the $E_{7(7)}$ quartic invariant of the microscopic charges of the ring [14], equation (2.17) implies that the microscopic angular momentum of the ring is:

$$J_T = J_\Delta + n^I N_I^{\text{BH}} = \frac{R^2 L^4}{l_p^6} \left(\sum n^I \right) + n^I N_I^{\text{BH}}. \quad (2.18)$$

Hence, the angular momenta of the solution may be re-written in terms of fundamental charges as:

$$\begin{aligned} J_1 &= J_T + \left(\frac{1}{6} C_{IJK} n^I n^J n^K + \frac{1}{2} \bar{N}_I n^I + J^{\text{BMPV}} \right) \\ J_2 &= - \left(\frac{1}{6} C_{IJK} n^I n^J n^K + \frac{1}{2} \bar{N}_I n^I + N_I^{\text{BH}} n^I + J^{\text{BMPV}} \right) \end{aligned} \quad (2.19)$$

Notice that in this form, J_1 contains no contribution coming from the combined effect of the electric field of the black hole and the magnetic field of the black ring. Such a contribution only appears in J_2 .

In an adiabatic process in which one moves the black hole into the center of a black ring one will need to understand how to keep the black ring (and black hole) “the same” during the process. Obviously the quantized charges N_I , N_I^{BH} and n^I , which can be measured on suitable Gaussian surfaces [19], must remain unchanged. Furthermore, because of the connection with the microscopic charges of the four-dimensional black hole and the $E_{7(7)}$ invariant, one might expect J_T to remain unchanged as well. This leads one to expect that the only thing that could change is J_2 , and only through the term that represents the contribution from the combined effect of the electric field of the black hole and the magnetic field of the black ring. We will show in section 5 that this intuitive expectation is precisely born out by the exact solution.

3. Obtaining the new solution

3.1 Setting up the problem

The dipole fluxes, Θ^I , are determined solely by the position of the ring and so remain unchanged. The harmonic functions, Z_I , are sourced by the M2 branes and therefore when we move the black hole off-center we must make a simple translation of the source, (2.9), that corresponds to the black hole. The non-trivial consequence of this lies in the third step of the linear algorithm where one solves for the angular momentum vector: There is a more complicated contribution generated by the electric field of the brane interacting with the magnetic field of the ring.

If we locate a black hole at $(r, \phi, z, \psi) = (a, 0, b, 0)$, then the Euclidean distance, d^2 , from a generic point, (r, ϕ, z, ψ) , to the black hole is given by:

$$d^2 = (r^2 + a^2 - 2ar \cos \phi) + (z^2 + b^2 - 2bz \cos \psi) = -\frac{R^2}{(x-y)} \rho \quad (3.1)$$

where

$$\rho \equiv (1 + \alpha^2 + \beta^2) y + (1 - \alpha^2 - \beta^2) x + 2\alpha \sqrt{1 - x^2} \cos \phi + 2\beta \sqrt{y^2 - 1} \cos \psi, \quad (3.2)$$

and $\alpha \equiv \frac{a}{R}$, $\beta \equiv \frac{b}{R}$. Note that $\rho \leq 0$. For a single black hole with charges, Y_I , we will orient the coordinates so that $\alpha, \beta \geq 0$, and take:

$$H_I = -\frac{Y_I(x-y)}{R^2 \rho}. \tag{3.3}$$

This is simply the translation of (2.9) to the new center: $(a, 0, b, 0)$. For the present we will make no specific assumptions about the functions, H_I , except that they vanish at infinity and are regular on the black ring.

The H_I now generate another source term on the right-hand side of the third equation in (2.3). We therefore define \hat{k} by:

$$k_\psi = (y^2 - 1) \left(\frac{1}{3} C(x+y) + \frac{1}{2} B \right) - A(y+1) + \hat{k}_\psi, \tag{3.4}$$

$$k_\phi = -(1-x^2) \left(\frac{1}{3} C(x+y) + \frac{1}{2} B \right) + \hat{k}_\phi, \tag{3.5}$$

$$k_x = \hat{k}_x, \quad k_y = \hat{k}_y, \tag{3.6}$$

where A, B , and C have been defined in (2.11), and \hat{k} satisfies:

$$d\hat{k} + *d\hat{k} = 2q^I H_I (dx \wedge d\phi - dy \wedge d\psi). \tag{3.7}$$

The expression for $k - \hat{k}$ is simply the angular momentum vector of the known black-ring solution.

The boundary conditions for \hat{k} are determined first by requiring that it is non-singular, except possibly at the black ring and at the black hole, and then by requiring that there are no closed time-like curves (CTC's) in the solution. In practice the latter is hard to establish globally, and we will not do it here, but there is a significant danger of getting CTC's near the ring and near the black hole, and it is by removing these CTC's that we fix the final boundary conditions on \hat{k} .

As discussed in [6], adding angular momentum to a black hole corresponds to adding a homogeneous solution of the third equation in (2.3). This homogeneous solution is centered on the black hole, which is now away from the center of the coordinate system. By shifting the homogeneous solution in (2.10) to $r = a$ and re-expressing it in the $x - y$ coordinate system we obtain:

$$\begin{aligned} \hat{k}_x^{BMPV} &= \frac{\alpha K(xy-1) \sin \phi}{R^2 \sqrt{1-x^2} \rho^2}, & \hat{k}_y^{BMPV} &= \frac{\alpha K \sqrt{1-x^2} \sin \phi}{R^2 \rho^2} \\ \hat{k}_\psi^{BMPV} &= \frac{K(y^2-1)}{R^2 \rho^2}, & \hat{k}_\phi^{BMPV} &= \frac{K[x^2-1 + \alpha(x-y)\sqrt{1-x^2} \cos \phi]}{R^2 \rho^2} \end{aligned} \tag{3.8}$$

Since the equations determining k are linear, and k^{BMPV} satisfies (2.3) by construction, we will put it aside for now, and add it to the final solution at the very end.

Finally, the system the equations for k is gauge invariant. Note that a gauge transformation, $k \rightarrow k + df$, for some function, f , of the spatial coordinates, is equivalent to the coordinate change, $t \rightarrow t + f$. Thus gauge fixing amounts to adjusting the rotational

behavior of the coordinate system. There are several natural choices for the gauge, but we will typically choose $k_y = 0$ because it simplifies the analysis of the metric near the ring. Also note that we may need to impose some boundary conditions on the gauge choice so as to make sure that the surfaces of constant time are a non-rotating frame at infinity.

The explicit system of equations that we have to solve is:

$$\begin{aligned} (y^2 - 1)(1 - x^2)(\partial_y \hat{k}_x - \partial_x \hat{k}_y) + (\partial_\psi \hat{k}_\phi - \partial_\phi \hat{k}_\psi) &= 0, \\ (y^2 - 1)(\partial_y \hat{k}_\phi - \partial_\phi \hat{k}_y) + (1 - x^2)(\partial_x \hat{k}_\psi - \partial_\psi \hat{k}_x) &= 0, \\ (\partial_y \hat{k}_\psi - \partial_\psi \hat{k}_y) + (\partial_\phi \hat{k}_x - \partial_x \hat{k}_\phi) &= -2q^I H_I. \end{aligned} \quad (3.9)$$

3.2 The solution for a vertically displaced black hole

In this paper we will focus upon the configuration depicted in figure 1, that is, solutions with vertically displaced black holes in which the U(1) symmetry of the plane of the ring is preserved, but the U(1) symmetry in the (r, ϕ) -plane is broken. Hence, we will take $b = \beta = 0$ and so (3.2) reduces to:

$$\rho \equiv (1 + \alpha^2)y + (1 - \alpha^2)x + 2\alpha\sqrt{1 - x^2}\cos\phi. \quad (3.10)$$

The solution we seek must therefore be independent of the angle ψ . If we now go to the gauge with $k_y = 0$, one can eliminate between the equations of (3.9) to show that:

$$(y^2 - 1)\partial_y^2 \hat{k}_\psi + \frac{1}{(1 - x^2)}\partial_\phi^2 \hat{k}_\psi + \partial_x((1 - x^2)\partial_x \hat{k}_\psi) = -2q^I(y^2 - 1)\partial_y H_I. \quad (3.11)$$

Using this equation, and the form of the original solution, (2.10), (with $\alpha = 0$), it is not very difficult to find a particular solution for the source term in (3.11). The more subtle issue is the careful choice of the homogeneous solution in (3.11). Indeed, the following is a homogeneous solution for all values of the constants a_j :

$$\hat{k}_\psi^{(0)} = a_1 + a_2 y + a_3 \rho^{-1}((1 + \alpha^2)x + (1 - \alpha^2)y). \quad (3.12)$$

The correct admixture of this with the particular solution is determined by the boundary conditions in all components of k . The solution we want is:

$$\begin{aligned} k_\psi &= (y^2 - 1)\left(\frac{1}{3}C(x + y) + \frac{1}{2}B\right) - A(y + 1) \\ &\quad - \frac{D}{2(1 + \alpha^2)R^2}(y + 1)\left[1 - \frac{1}{\rho}((1 + \alpha^2)(x - y) + 2)\right], \end{aligned} \quad (3.13)$$

Given this one can now integrate (3.9) to obtain the other components of \hat{k} . To express the final result it is useful to define:

$$\sigma \equiv (1 + \alpha^2)y + (1 - \alpha^2)x + 2\alpha\sqrt{1 - x^2}\cos\phi \quad (3.14)$$

and introduce the function:

$$F(x, y) \equiv ((1 + \alpha^2)x + (1 - \alpha^2)y)\left(\frac{1}{\sigma\rho} + \frac{1}{\sigma^2}\log\left(1 - \frac{\sigma}{\rho}\right)\right). \quad (3.15)$$

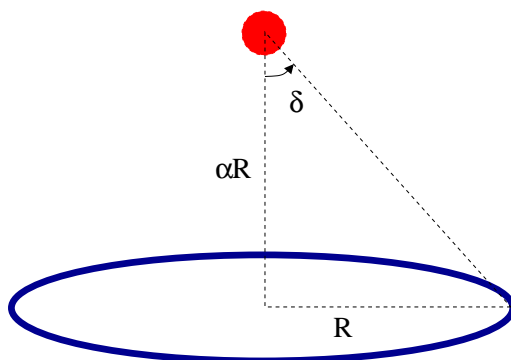


Figure 1: This shows the configuration of the black hole and black ring that is described by the new solution. The parameter, α , is related to the angle of approach, δ , by $\alpha \equiv \cot \delta$.

Then one has

$$\begin{aligned}
 k_\phi = & -(1-x^2) \left(\frac{1}{3} C(x+y) + \frac{1}{2} B \right) - \frac{D(1-x^2)}{2R^2} \frac{1}{\sigma} \log \left(1 - \frac{\sigma}{\rho} \right) \\
 & + \frac{D(1-x^2)}{2(1+\alpha^2)R^2} \left((1-\alpha^2) - \frac{2\alpha x \cos \phi}{\sqrt{1-x^2}} \right) \left(F(x,y) + \frac{1}{\rho} \right) \\
 & + \frac{D \log(1+\alpha^2)}{2\alpha(1+\alpha^2)R^2} \sqrt{1-x^2} \cos \phi + \frac{D(\alpha^2 - \log(1+\alpha^2))}{2\alpha^2(1+\alpha^2)R^2} (1-x^2) \cos 2\phi, \quad (3.16)
 \end{aligned}$$

$$\begin{aligned}
 k_x = & \frac{\alpha D \sin \phi}{(1+\alpha^2)R^2 \sqrt{1-x^2}} \left(\frac{1}{\rho} + F(x,y) \right) - \frac{D \log(1+\alpha^2)}{2\alpha(1+\alpha^2)R^2} \frac{x}{\sqrt{1-x^2}} \sin \phi \\
 & - \frac{D(\alpha^2 - \log(1+\alpha^2))}{2\alpha^2(1+\alpha^2)R^2} x \sin 2\phi. \quad (3.17)
 \end{aligned}$$

In integrating to obtain \hat{k} there are some important integration “constants” and gauge ambiguities to be resolved. This is done first by making sure that angular momentum vector, k , is regular everywhere except at the black hole ($\rho = 0$) and at the black ring ($y = -\infty$). The last two terms in k_ϕ and in k_x are, in fact, pure gauge and could, in principle, be discarded. Gauge transformations in k do, however, amount to the choice of the spatial sections of the metric (the surfaces of constant t) and the inclusion of these particular gauge terms in k amounts to the choice of a non-rotating coordinate system at infinity.

3.3 Regularity

Looking at the form of k , it appears that there might be singularities at $\sigma = 0$, but if one expands the logarithms for small σ one easily sees that k is smooth at $\sigma = 0$. The logarithms may also be re-written as $\log((1+\alpha^2)(y-1)/\rho)$ and, away from the black hole and black ring, the argument is positive definite because ρ is negative definite and $-\infty < y < -1$.

The angular momentum component, k_x , is singular at $x = \pm 1$, both in (3.8) and in the new solution, but this singularity is a coordinate artifact because the (x, ϕ) coordinate system is degenerate at these points. Obviously, \hat{k}^{BMPV} cannot be singular at $x = \pm 1$ because it is just the translation of a regular vector field to a different coordinate system

To show that k is non-singular at $x = \pm 1$, one first collects all the terms involving $\sqrt{1-x^2}$:

$$\begin{aligned} \tilde{k} = & \frac{\alpha D}{(1+\alpha^2)R^2} \left(\frac{\sin\phi}{\sqrt{1-x^2}} dx - x\sqrt{1-x^2} \cos\phi d\phi \right) \left(F(x,y) + \frac{1}{\rho} \right) \\ & + d \left(\frac{D \log(1+\alpha^2)}{2\alpha(1+\alpha^2)R^2} \sqrt{1-x^2} \sin\phi \right). \end{aligned} \quad (3.18)$$

Now introduce the Cartesian coordinates: $u = \sqrt{1-x^2} \cos\phi$, $v = \sqrt{1-x^2} \sin\phi$, which are regular near $x = \pm 1$. Indeed, one has:

$$du^2 + dv^2 = \frac{x^2 dx^2}{(1-x^2)} + (1-x^2) d\phi^2, \quad (3.19)$$

which is conformal to the (x, ϕ) part of (2.6) near $x = \pm 1$. Now observe that:

$$\begin{aligned} \tilde{k} = & -\frac{\alpha D}{(1+\alpha^2)R^2} \left(F(x,y) + \frac{1}{\rho} \right) x dv + d \left(\frac{D \log(1+\alpha^2)}{2\alpha(1+\alpha^2)R^2} v \right) \\ & + \frac{\alpha D}{(1+\alpha^2)R^2} \left(F(x,y) + \frac{1}{\rho} \right) v dx, \end{aligned} \quad (3.20)$$

which means that k is completely regular at $u = v = 0$, or $x = \pm 1$. One can also do a gauge transformation that makes k manifestly regular at $x = \pm 1$. This is done in the appendix.

3.4 Horizons

The analysis of the horizon of a black hole is almost trivial. Near the black hole, the functions, Z_I , and the warp factor in (2.1) behave as:

$$Z_I \sim \frac{Y_I}{d^2} \quad \Rightarrow \quad (Z_1 Z_2 Z_3)^{1/3} \sim \frac{(Y_1 Y_2 Y_3)^{1/3}}{d^2}, \quad (3.21)$$

Thus, for a non-rotating black hole, the three-spheres around the black hole (at $d = 0$) limit to a sphere of constant radius of order $(Y_1 Y_2 Y_3)^{1/3}$. (If the black hole rotates then there are terms that contribute at the same order coming from the angular momentum vector, \hat{k}^{BMPV} .) The terms in Z_I that are relevant for the horizon geometry are all $\mathcal{O}(d^{-2})$, while nearby BPS objects only modify the Z_I at terms of $\mathcal{O}(d^0)$ as $d \rightarrow 0$. Therefore, the horizon geometry of the black hole is completely oblivious to any other BPS objects nearby.

Conversely, if one carefully examines the derivation of the near-ring metric one sees that leading powers of y in k_ψ are determined by requiring them to cancel against the *leading and first sub-leading* orders of divergence in the warp-factor, $(Z_1 Z_2 Z_3)^{1/3}$. (Remember that the ring is located at $y = -\infty$.) Terms at the second sub-leading order then generate *finite* corrections to the near-ring geometry. Thus constant terms in the Z_I and terms of order y in k_ψ can only make a finite contribution to the near horizon geometry. This means that the nearby black hole will not cause any new singular behavior near the ring horizon, but the horizon geometry can, and indeed does, have a finite response to such nearby BPS objects. The horizon area of the deformed black ring will be given in section 5.

The fact that the entropy density depends on the value of the black-hole electric fields at the horizon, means that a black-hole/black-ring solution in which the U(1) along the ring is not preserved will have a ring horizon whose area changes as one goes along the ring. Such configurations are very instructive to study, and are currently under examination. Here, however, we focus on the processes that preserve the U(1) symmetry of the ring.

4. The complete solution

The metric and forms are given by equations (2.1) and (2.2), with

$$\Theta^J = 2 q^J (dx \wedge d\phi - dy \wedge d\psi), \quad (4.1)$$

and

$$Z_I = 1 + \frac{\bar{Q}_I}{R}(x-y) - \frac{2 C_{IJK} q^J q^K}{R^2}(x^2 - y^2) - \frac{Y_I(x-y)}{R^2 \rho}, \quad (4.2)$$

where

$$\rho = (1 + \alpha^2)y + (1 - \alpha^2)x + 2\alpha \sqrt{1 - x^2} \cos \phi, \quad (4.3)$$

and $\alpha \equiv \frac{a}{R}$. To express the angular momentum vectors we introduce the quantities

$$\begin{aligned} \sigma &\equiv (1 + \alpha^2) + (1 - \alpha^2)x + 2\alpha \sqrt{1 - x^2} \cos \phi, \\ F(x, y) &\equiv ((1 + \alpha^2)x + (1 - \alpha^2)) \left(\frac{1}{\sigma \rho} + \frac{1}{\sigma^2} \log \left(1 - \frac{\sigma}{\rho} \right) \right). \end{aligned} \quad (4.4)$$

The total angular momentum vector is obtained by adding the solution obtained in the previous section and the homogeneous solution corresponding to the rotation of the black hole (3.8). One obtains:

$$\begin{aligned} k_\psi &= (y^2 - 1) \left(\frac{1}{3} C(x+y) + \frac{1}{2} B \right) - A(y+1) \\ &\quad - \frac{D}{2(1+\alpha^2)R^2} (y+1) \left[1 - \frac{1}{\rho} ((1+\alpha^2)(x-y) + 2) \right] + \frac{K(y^2-1)}{R^2 \rho^2}, \\ k_\phi &= -(1-x^2) \left(\frac{1}{3} C(x+y) + \frac{1}{2} B \right) - \frac{D(1-x^2)}{2R^2} \frac{1}{\sigma} \log \left(1 - \frac{\sigma}{\rho} \right) \\ &\quad + \frac{D(1-x^2)}{2(1+\alpha^2)R^2} \left((1-\alpha^2) - \frac{2\alpha x \cos \phi}{\sqrt{1-x^2}} \right) \left(F(x, y) + \frac{1}{\rho} \right) \\ &\quad + \frac{D \log(1+\alpha^2)}{2\alpha(1+\alpha^2)R^2} \sqrt{1-x^2} \cos \phi + \frac{D(\alpha^2 - \log(1+\alpha^2))}{2\alpha^2(1+\alpha^2)R^2} (1-x^2) \cos 2\phi \\ &\quad + \frac{K \left[x^2 - 1 + \alpha(x-y)\sqrt{1-x^2} \cos \phi \right]}{R^2 \rho^2}, \\ k_x &= \frac{\alpha D \sin \phi}{(1+\alpha^2)R^2 \sqrt{1-x^2}} \left(\frac{1}{\rho} + F(x, y) \right) - \frac{D \log(1+\alpha^2)}{2\alpha(1+\alpha^2)R^2} \frac{x}{\sqrt{1-x^2}} \sin \phi \\ &\quad - \frac{D(\alpha^2 - \log(1+\alpha^2))}{2\alpha^2(1+\alpha^2)R^2} x \sin 2\phi + \frac{\alpha K(xy-1) \sin \phi}{R^2 \sqrt{1-x^2} \rho^2}, \\ k_y &= \frac{\alpha K \sqrt{1-x^2} \sin \phi}{R^2 \rho^2}. \end{aligned} \quad (4.5)$$

The quantities A, B, C, D are defined in (2.11), and the relations between the black ring and black hole microscopic charges and the parameters appearing in the solution are given in (2.12).

5. The charges of the new solution

The asymptotic charges of the new solutions are the same as those of the solution with the black hole in the center of the ring:

$$N_I = \bar{N}_I + N_I^{\text{BH}} + \frac{1}{2} C_{IJK} n^J n^K. \quad (5.1)$$

The angular momenta of the solution can easily be obtained by expanding (3.13) and (3.16) near spatial infinity. One finds:

$$\begin{aligned} J_1 &= J_\Delta + \left(\frac{1}{6} C_{IJK} n^I n^J n^K + \frac{1}{2} \bar{N}_I n^I + \frac{N_I^{\text{BH}} n^I}{1 + \alpha^2} + J^{\text{BMPV}} \right) \\ J_2 &= - \left(\frac{1}{6} C_{IJK} n^I n^J n^K + \frac{1}{2} \bar{N}_I n^I + \frac{N_I^{\text{BH}} n^I}{1 + \alpha^2} + J^{\text{BMPV}} \right) \end{aligned} \quad (5.2)$$

where J_Δ is the same as in the concentric configuration:

$$J_\Delta \equiv \frac{R^2 L^4}{l_p^6} \left(\sum n^I \right). \quad (5.3)$$

One can also find the horizon area of the ring, and read off its entropy:

$$S = \frac{2\pi A}{\kappa_{11}^2} = \pi \sqrt{\mathcal{M}}, \quad (5.4)$$

where now

$$\begin{aligned} \mathcal{M} \equiv & 2n^1 n^2 \bar{N}_1 \bar{N}_2 + 2n^1 n^3 \bar{N}_1 \bar{N}_3 + 2n^2 n^3 \bar{N}_2 \bar{N}_3 - (n^1 \bar{N}_1)^2 - (n^2 \bar{N}_2)^2 - (n^3 \bar{N}_3)^2 \\ & - 4n^1 n^2 n^3 \left(J_\Delta + \frac{n^I N_I^{\text{BH}}}{1 + \alpha^2} \right). \end{aligned} \quad (5.5)$$

As we explained in section 3, the presence of terms proportional to N_I^{BH} in \mathcal{M} indicates that the horizon of the black ring “feels” the presence of the black hole. In contrast, the black hole horizon is completely insensitive to the presence of other BPS objects nearby.

Again, since the entropy of the black ring is the square root of the $E_{7(7)}$ quartic invariant of the microscopic charges of the ring [14], equation (5.5) implies that the microscopic angular momentum of the ring is:

$$J_T = J_\Delta + \frac{n^I N_I^{\text{BH}}}{1 + \alpha^2} \equiv \frac{R^2 L^4}{l_p^6} \left(\sum n^I \right) + \frac{n^I N_I^{\text{BH}}}{1 + \alpha^2}. \quad (5.6)$$

When written in terms of the microscopic charges of the ring, the angular momenta of the solution become:

$$\begin{aligned} J_1 &= J_T + \left(\frac{1}{6} C_{IJK} n^I n^J n^K + \frac{1}{2} \bar{N}_I n^I + J^{\text{BMPV}} \right) \\ J_2 &= - \left(\frac{1}{6} C_{IJK} n^I n^J n^K + \frac{1}{2} \bar{N}_I n^I + \frac{N_I^{\text{BH}} n^I}{1 + \alpha^2} + J^{\text{BMPV}} \right). \end{aligned} \quad (5.7)$$

Our solutions describe black rings with arbitrary charges, dipole charges and angular momenta, and black holes with arbitrary charges and angular momenta. However, if we want to study the adiabatic merger of a black hole and a black ring, we have to focus on a subset of our solutions that describe *the same* black hole and *the same* black ring at arbitrary separations.

This implies that the charges N_I^{BH} of the black hole and the charges \bar{N}_I and dipole charges n^I of the black ring must remain the same. The other quantity that must remain invariant is J_T , both because it is a microscopic charge of the black ring, and because in an adiabatic process the total area of all horizons must remain constant. Since the black-hole horizon remains unchanged, the ring horizon area, (5.5), must therefore remain constant. The fact that \bar{N}_I and n^I are the same then automatically implies that in an adiabatic process J_T does not change.

This implies that the angular momentum, J_1 , also remains constant as one moves the black ring and black hole by adiabatically varying α . This is to be expected since the ψ translations remain a symmetry of the solution throughout this process and so J_1 must be conserved.

On the other hand, the angular momentum component, J_2 , depends on the relative position of the ring and of the black hole. The relevant term, $\frac{N_I^{\text{BH}} n^I}{1+\alpha^2}$, comes from the Poynting vector sourced by the magnetic fields of the ring and the electric fields of the black hole. When the black ring is infinitely far away, one has $\alpha \rightarrow \infty$, and this term vanishes.

It is possible to see intuitively why the angular momentum coming from fluxes changes as the black hole and the ring move apart. In four dimensions both magnetic and electric charges are point-like, and the angular momentum coming from having an electron and a magnetic monopole is the same regardless of their relative position. Similarly, in five dimensions the angular momentum coming from the magnetic field of an infinitely long black string and the electric field of a black hole should be constant. However, if we have a black *ring* and a black hole, the magnetic field of the former decays faster than that of an infinitely long string, and hence the angular momentum coming from the integral of $\vec{E} \times \vec{B}$ should depend on the distance between the two objects.

The constancy of J_T along with (5.6) imply that as the black hole is brought near the black ring, the embedding radius of the latter, R , must change according to:

$$R^2 = \frac{l_p^6}{L^4} \left(\sum n^I \right)^{-1} \left(J_T - \frac{n^I N_I^{\text{BH}}}{1 + \alpha^2} \right). \tag{5.8}$$

For fixed microscopic charges this formula gives the radius of the ring as a function of the parameter α .

If one of the charges and two of the dipole charges of the black ring are set to zero, it becomes a two-charge supertube [20]. The physics of a supertube probe in a BMPV black hole background has been studied before using the Born-Infeld action of a supertube [3, 11]. The supertube analysis gives a nontrivial check to two of the the phenomena we observe. The first is that one cannot move the black hole off the center of the ring without changing J_2 [11]. The second is that the supertube radius changes as the black hole is moved.

In the supertube limit ($\overline{N}_3, n^1, n^2 \rightarrow 0$) the radius formula (5.8) reproduces exactly the radius of the two charge supertube in the three-charge black hole background computed in [3, 11].

6. Mergers and acquisitions

We have seen that as we vary the separation of the black hole and the black ring, the embedding radius of the ring changes according to (5.8). Before we begin a detailed discussion of this, we want to underline that even though the embedding radius varies, the physical size of the ring, as determined by that of the horizon of the black ring, will remain constant. Thus requiring the process to be both adiabatic and retain the U(1) symmetry of the ring means that the ring will remain rigid.

6.1 The merging process

Equation (5.8) shows that for certain values of the charges, the ring radius, R , goes to zero at a non-zero value of α . One should remember that the distance (in \mathbb{R}^4) between the black hole and the plane of the ring is αR , and so the limit $R \rightarrow 0$ not only means that the embedding radius is vanishing but also that the black hole is limiting to the ring plane. Hence, when $R \rightarrow 0$ the black ring is merging with the black hole. The value of α where this happens simply determines the black-hole latitude at which the ring arrives, or “crowns,” the black hole. (See figure 2.)

The physics here is similar to what happens to supertubes and black rings in Taub-NUT [21, 22, 16, 15] as one changes the Taub-NUT radius. The physical size of the ring is completely fixed by its charges. If the size of the space is modified (either by bringing in a black hole, or by modifying the Taub-NUT radius) the ring migrates to a different location, so as to maintain its actual physical size.

Even though the black hole appears point-like in the embedding space, it has a physical size determined by its charges. Indeed, it is worth recalling that, in \mathbb{R}^4 polar coordinates centered on the black hole (with $r = d \sin \chi$, $z = d \cos \chi$), the horizon metric of the black hole is:

$$ds_3^{BH} = (Y_1 Y_2 Y_3)^{1/3} \left(d\chi^2 + \sin^2 \chi d\phi^2 + \cos^2 \chi d\psi^2 - \frac{K^2}{Y_1 Y_2 Y_3} (\sin^2 \chi d\phi - \cos^2 \chi d\psi)^2 \right). \quad (6.1)$$

As one can see from (5.8), whether a ring and a black hole merge or not, is completely independent of the black hole angular momentum. This is perhaps somewhat surprising, particularly given the fact that a maximally-spinning black hole has zero horizon area; however one can see from (6.1) that the circles in the ψ -direction, over which the ring must slip, have a radius that is bounded below (at $\chi = \frac{\pi}{4}$) by $\frac{1}{2}(Y_1 Y_2 Y_3)^{1/6}$, which is independent of K . This observation is, by no means, a complete explanation, particularly when the ring approaches the black hole from very close to the equatorial plane. It would be interesting to see if one can understand (5.8) entirely from the geometry of “black hole hoopla.”

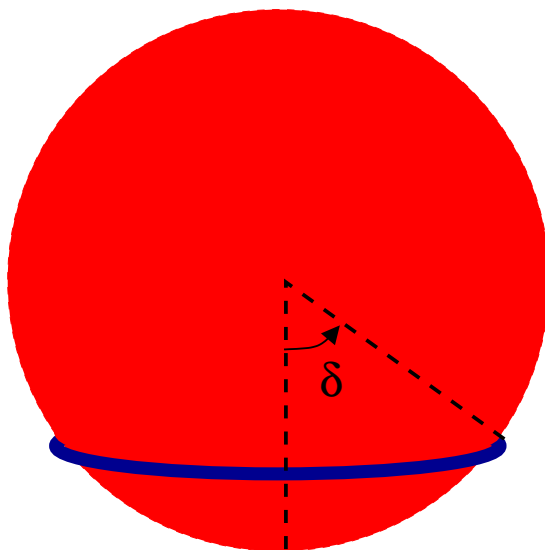


Figure 2: This shows the black hole and the black ring at the merger. The ring has $R = 0$. The value of α at the merger gives the latitude at which the ring crosses the horizon ($\alpha \equiv \cot \delta$).

If the charges are such that R never becomes zero as α goes to zero, then, as one passes the black hole through the center of the black ring, the embedding radius just shrinks to a minimal size, and then re-expands to its “normal” radius when the black hole moves infinitely far away.

Equation (5.7) implies that in order to bring the black ring to the black-hole horizon, one has to increase the angular momentum J_2 . For the merger to happen one must have $R \rightarrow 0$ for some value of α . This means that $J_\Delta = 0$, and from (5.2) we see that this means $J_1 = J_2$. Thus the black object that results from the merger has charges, N_I , given in (5.1) and angular momenta both equal to:

$$J_{\text{final}}^{\text{BMPV}} = J_T + \frac{1}{6} C_{IJK} n^I n^J n^K + \frac{1}{2} \overline{N}_I n^I + J^{\text{BMPV}}, \tag{6.2}$$

and is nothing but a BMPV black hole. Hence, we have shown that to merge a black ring with a three-charge black hole in this axially symmetric manner, one has to add enough angular momentum to the solution so that the resulting black hole has equal angular momenta.

This result solves an old puzzle. When probing supersymmetric black holes with supertubes, it was found that a supertube can be easily dumped into a black hole [3, 11]. Since supertubes have generically unequal angular momenta, the resulting supersymmetric black hole would have naively had unequal angular momenta. However, such supersymmetric black holes were argued not to exist at all in five-dimensional minimal supergravity [10]. The key to the solution of this puzzle is the fact that the total angular momentum coming from fluxes changes as the supertube moves towards the black hole. When the two merge, the angular momentum coming from fluxes exactly equals the supertube/ring angular momentum, and the resulting object has equal angular momenta.

6.2 Chronology protection and the entropy of mergers

As discussed above, whether a ring and a black hole merge or not, is completely independent of the angular momentum of the black hole. Thus, one can consider a black hole that is maximally spinning and merge it with a black ring. One can also give the black ring a very high angular momentum, by choosing a J_T that renders the ring entropy very close to zero.

The resulting object will have a rather large angular momentum, and it is interesting to see if this angular momentum can be larger than the maximal allowed angular momentum of the resulting black hole, $J_{\max} = \sqrt{N_1 N_2 N_3}$. A related question is whether the entropy of the black hole that results from the merger is larger than the sum of the entropies of the ring and hole that merge. Since the entropy of the BMPV black hole is

$$S_{BH} = 2\pi \sqrt{N_1 N_2 N_3 - J^2} \tag{6.3}$$

an entropy increase always implies chronology protection, but the reverse is not necessarily true.

The general problem of finding whether black hole mergers with black rings are thermodynamically irreversible involves eleven parameters: n^I , N_I , N_I^{BH} , R and $J_{\text{initial}}^{BMPV}$, and it is rather too involved to analyze completely here. To simplify the algebra we consider a representative five-parameter sub-class of solutions in which we set:

$$n^I = n, \quad \bar{N}_I = \bar{N}, \quad N_I^{BH} = N, \quad I = 1, 2, 3. \tag{6.4}$$

We consider a ring starting from infinity with $R = R_\infty$ and being adiabatically merged with a BMPV black hole of initial angular momentum, $J_{\text{initial}}^{BMPV} = J$.

The final state must be a BMPV black hole with charges given by (5.1):

$$N_{\text{final}} = N + \bar{N} + n^2, \tag{6.5}$$

and with final angular momenta given by (6.2):

$$J_{\text{final}} = 3(L^4 l_p^{-6}) R_\infty^2 n + n^3 + \frac{3}{2} \bar{N} n + J. \tag{6.6}$$

The change of entropy in the merger process is then:

$$\Delta S = 2\pi \sqrt{N_{\text{final}}^3 - J_{\text{final}}^2} - \left(\pi \sqrt{\mathcal{M}} + 2\pi \sqrt{N^3 - J^2} \right), \tag{6.7}$$

where \mathcal{M} is given by (2.17). To show that this is non-negative, it is equivalent to show that the function:

$$\mathcal{G}(n, \bar{N}, N, R_\infty, J) \equiv (N_{\text{final}}^3 - J_{\text{final}}^2) - \left(\frac{1}{2} \sqrt{\mathcal{M}} + \sqrt{N^3 - J^2} \right)^2 \tag{6.8}$$

is non-negative.

We will, of course, require that the initial states have non-negative horizon areas:

$$\mathcal{M} = 3n^2 \left(\bar{N}^2 - 4n^2 (L^4 l_p^{-6}) R_\infty^2 \right) \geq 0 \quad \Leftrightarrow \quad \bar{N} \geq 2n (L^2 l_p^{-3}) R_\infty, \tag{6.9}$$

$$N^3 - J^2 \geq 0. \tag{6.10}$$

There is also the condition that the merger actually happens. We know that J_T is constant and so

$$J_T = \frac{R^2 L^4}{l_p^6} \left(\sum n^I \right) + \frac{n^I N_I^{\text{BH}}}{1 + \alpha^2} = \frac{R_\infty^2 L^4}{l_p^6} \left(\sum n^I \right). \quad (6.11)$$

Therefore, in order for $R \rightarrow 0$ at some value of α one must have:

$$(L^4 l_p^{-6}) R_\infty^2 \leq N. \quad (6.12)$$

Therefore our task is to show that \mathcal{G} is non-negative in the domain defined by (6.9), (6.10) and (6.12).

First consider the dependence on J . One can easily check that $\frac{d\mathcal{G}}{dJ} < 0$ at $J = 0$ and $\frac{d\mathcal{G}}{dJ} \rightarrow +\infty$ as $J \rightarrow N^{3/2}$, and so, perhaps rather surprisingly, \mathcal{G} , is not minimized at $J = N^{3/2}$. The actual minimum, as a function of J occurs at:

$$J = \frac{N^{3/2} (n^2 + \frac{3}{2} \bar{N} + 3(L^4 l_p^{-6}) R_\infty^2)}{\sqrt{(n^2 + \frac{3}{2} \bar{N} + 3(L^4 l_p^{-6}) R_\infty^2)^2 + \frac{3}{4} (\bar{N}^2 - 4n^2 (L^4 l_p^{-6}) R_\infty^2)}}, \quad (6.13)$$

which clearly lies in the range $0 < J \leq N^{3/2}$ and hits the upper bound if and only if the black ring is actually a supertube with $\mathcal{M} = 0$.

Now let \mathcal{G}_1 be the function \mathcal{G} evaluated at the minimizing value of J in (6.13). Furthermore, consider \mathcal{G}_1 as a function of R_∞ . One can easily check that $\frac{d\mathcal{G}_1}{dR_\infty} < 0$ and so the minimum of \mathcal{G}_1 occurs at the maximum value of R_∞ . We have two bounds on R_∞ given by (6.9) and (6.12). The former is the relevant limit if $N \geq \frac{\bar{N}^2}{4n^2}$ and the latter is relevant for $N \leq \frac{\bar{N}^2}{4n^2}$.

Suppose that the bound in (6.12) is saturated; then, from (6.9), one has $\bar{N}^2 \geq 4n^2 N$ and one can also show that:

$$\mathcal{G}_1 = p_2 - \sqrt{p_2^2 - (\bar{N}^2 - 4n^2 N) p_1^2}, \quad (6.14)$$

where

$$p_1 \equiv \bar{N}^2 + 3N\bar{N} + 3N^2 - n^2 N, \quad p_2 \equiv \bar{N} p_1 - 2n^2 N(3N + \bar{N}). \quad (6.15)$$

Since $\bar{N}^2 \geq 4n^2 N$, one has $p_1 > 0$, and so $\mathcal{G}_1 \geq 0$ with equality if and only if $\bar{N}^2 = 4n^2 N$. That is, $\mathcal{G}_1 \geq 0$ with $\mathcal{G}_1 = 0$ if and only if *both* bounds, (6.12) and (6.9), are saturated.

Now suppose that the bound in (6.9) is saturated and introduce the variable $\nu = \sqrt{\bar{N}}$. From (6.12), one has $\bar{N} \leq 2n\nu$. One can now show that:

$$\begin{aligned} \mathcal{G}_1 = \frac{1}{16n^2} (2n\nu - \bar{N}) (9\bar{N}^3 + 18n\bar{N}^2\nu + 12n^4\bar{N} + 20n^2\bar{N}^2) \\ + 8n^3\nu (3n^2 - 2n\nu + 3\nu^2) + 4n\nu\bar{N} (10n^2 - 3n\nu + 6\nu^2). \end{aligned} \quad (6.16)$$

Note that the quadratic forms that appear as coefficients of $8n^3\nu$ and $4n\nu$ are both strictly positive. Thus, in this limit we also have $\mathcal{G}_1 \geq 0$ with $\mathcal{G}_1 = 0$ if and only if *both* bounds, (6.12) and (6.9), are saturated.

Conversely, suppose that both bounds, (6.12) and (6.9), are saturated, then one finds:

$$\mathcal{G} = \frac{1}{2n} (N^{3/2} - J) (4n^4 + 6n^2 \bar{N} + 3\bar{N}^2). \quad (6.17)$$

It follows that the entropy increase is strictly positive unless *all three of the following are satisfied*: a) The black hole is maximally spinning, b) the black ring is maximally spinning (i.e. it has zero entropy) and c) the charge of the black hole is exactly the size needed for the black hole to only just capture the black ring ($R \rightarrow 0$ as $\alpha \rightarrow 0$). Hence, in the mergers with $\Delta S = 0$, the black ring must itself have zero entropy, and it has to settle on the equator of a maximally rotating black hole.

We therefore see that, at least for this sub-class of black holes and black rings, we not only have chronology protection, but that the merger process is, with one exception, thermodynamically irreversible. The only reversible merger matches with physical intuition: One must start with a black hole and a black ring both of zero entropy, and the black ring must merge by just grazing the black hole equator. While we have not analyzed the problem for the fully general three-charge objects, we expect chronology protection to work in exactly the same way.

Conversely, one can also investigate the condition for thermodynamically reversible mergers of fully-general black rings and black holes. In particular, suppose that one starts with a zero-entropy black ring and a maximally-rotating black hole, both with arbitrary charges, and one further imposes the physically sensible condition that the ring meets the black hole at the equator. For generic black holes and black rings we find that such a merger leads to a BMPV black hole of non-zero entropy. However, we also find that there are thermodynamically reversible mergers when the black ring and the black hole charges satisfy:

$$\bar{N}_I = \frac{\bar{P}}{n^I} \quad \text{and} \quad N_I^{BH} = \frac{P^{BH}}{n^I}, \quad (6.18)$$

for two integers, \bar{P} and P^{BH} . Also note that this means

$$N_I \equiv \bar{N}_I + \frac{1}{2} C_{IJK} n^J n^K = \frac{(\bar{P} + n^1 n^2 n^3)}{n^I}, \quad (6.19)$$

Thus the electric charges of black ring *and* its charges dissolved in fluxes ($\frac{1}{2} C_{IJK} n^J n^K$) must both be aligned exactly parallel to the electric charges of the black hole.

Our result suggests that the microstates corresponding to the maximally spinning BMPV black hole must belong to a special ensemble and that one cannot simply throw in any black ring microstate without rearranging the internal state of the BMPV black hole and thereby generating entropy. That is, to add a black ring microstate reversibly one has to precisely prepare this microstate so as to match the ensemble to which it is being added. We are presently investigating what this tells us about the ensemble of microstates that make up the BMPV black hole.

7. Conclusions and future directions

We have constructed smooth, five-dimensional supergravity solutions that describe a black ring and a black hole at an arbitrary distance from the center of the black ring. We have

found that as one moves the black ring towards the black hole the angular momentum in the plane of the ring remains constant, while the angular momentum in the plane perpendicular to the ring increases.

We have also analyzed the merger of a black ring with a black hole, and have shown that even if one starts with a solution where the two angular momenta are different, to bring the black ring into the black hole one has to change one of the angular momenta, such that after the merger the two angular momenta are equal.

In [11] it was argued that by throwing a supertube into an extremal (BPS) black hole one can create a non-extremal (non-BPS) black hole that is unstable. Our solutions show that *extremal* mergers (and presumably mergers that are very close to being extremal) can never produce an unstable object. Since supergravity solutions fully take into account back-reaction and angular momenta coming from fluxes, it would be very interesting to see whether one can use our solutions to learn more about the non-extremal mergers proposed in [11]. It would also be worthwhile to try to extend our work to the non-extremal solutions, and thus obtain a complete description of non-extremal mergers. A non-extremal extension of our solutions, even if only perturbative, will most certainly illuminate the physics of the merger process and allow one to understand how the resulting unstable black hole evolves.

It is also possible to construct solutions where the black hole is away from the center of the ring, but still in the plane of the ring. Such a solution would be very interesting, because the rotational invariance along the ring horizon would be broken (in contrast to the solutions that we have constructed here). Hence, the near-ring solution will probably look like the black rings with variable charge density constructed in [23]. If, as claimed in [19], such solutions are not smooth, we would have a process by which a smooth solution would become singular as one moves on a moduli space - this would be certainly very unexpected and interesting, and is currently under investigation.

Our configurations can be easily dualized to a frame in which the asymptotic charges are those of the D1-D5-P system. One can then take a near-horizon limit, and obtain an asymptotically $AdS_3 \times S^3 \times T^4$ solution, that contains a black hole and a black ring, and that is dual to an ensemble of boundary microstates. It would be very interesting to find what this ensemble is, and to give a microscopic description of the entropy of the black-ring black-hole configuration, similar to the microscopic description of the black rings in [14]. A way to attack this problem would be to compare the entropy of the ring-hole system to the entropy of a black ring with the same angular momenta and charges. If, for some values of the charges, the single-ring entropy is smaller than that of a ring-hole system, then we would have a very interesting phase diagram, that will most definitely improve our microscopic understanding of these objects. Another route of attack would be to use the fact that one can get different angular momenta using the same ring and the same black hole placed at different distances. If there exists any link between the ring dipole charges and the length of string bits in the boundary dual [14, 7, 24–26], then the fact then one obtains different angular momenta from the same objects might help clarify their microscopic descriptions.

Another interesting aspect of the mergers we have analyzed is the fact that a merger of a maximally spinning BMPV and a certain class of zero-entropy black rings appears to

be thermodynamically *reversible*, unlike the merger of say two BMPV black holes. This indicates that if one adds a certain zero-entropy black ring microstate to a microstate correspond to the maximally-spinning BMPV black hole, the result is another maximally-spinning BMPV microstate. Combining this with the knowledge of zero-entropy black ring microstates [12, 27] might allow us to find the microstates corresponding to the maximally-spinning BMPV black hole, which would significantly improve our understanding of black holes in string theory.

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A. Gauge transformations of the angular momentum vector

In our solution (3.17), the function k_x diverges like $\frac{1}{\sqrt{1-x^2}}$ near the points $x = 1$ and $x = -1$. Remembering that near the horizon $x = -\cos\theta$, this implies that the rotation vector k_θ is constant near $\theta = 0$ and $\theta = \pi$. To put the horizon metric in canonical form one can make a coordinate transformation, which eliminates this constant.

If the function k_x goes like $-\frac{a_1}{\sqrt{1-x}}$ at the south pole, and $\frac{a_2}{\sqrt{1+x}}$ at the north pole, then the gauge transformation

$$\vec{k} \rightarrow \vec{k} - \vec{\nabla} \left(\frac{(a_1(1+x) + a_2(1-x))\sqrt{1-x^2}}{\sqrt{2}} \right) \tag{A.1}$$

eliminates the divergent parts of k_x at the poles.

By expanding k_x we find that the function that enters the gauge transformation is

$$G \equiv \frac{D(1+x)\sqrt{1-x^2} \sin\phi}{2R^2\sqrt{1+\alpha^2}} \left[\frac{\log(\alpha^2+1)}{2\alpha} - \frac{\alpha}{2} \log \left(\frac{(y-1)(\alpha^2+1)}{y\alpha^2 - \alpha^2 + y + 1} \right) - \frac{2\alpha}{y\alpha^2 - \alpha^2 + y + 1} \right] + \frac{D(1-x)\sqrt{1-x^2} \sin\phi}{4\alpha R^2\sqrt{1+\alpha^2}} \left[\log(\alpha^2+1) - \log \left(\frac{(y-1)(\alpha^2+1)}{y\alpha^2 + \alpha^2 + y - 1} \right) \right] \tag{A.2}$$

and the new components of the angular momentum vector will be

$$\begin{aligned} k'_x &= k_x - \partial_x G, & k'_y &= -\partial_y G \\ k'_\psi &= k_\psi, & k'_\phi &= k_\phi - \partial_\phi G. \end{aligned} \tag{A.3}$$

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